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A BOUNDARY-LAYER THEORY FOR CELLULAR CONVECTION

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Abstract—A theory is developed for finite amplitude, steady cellular convection. The theory requires that the Prandtl number be large compared with unity and that the Rayleigh number be large compared with the critical Rayleigh number; only two-dimensional, laminar cells are considered. The core of each cell is an isothermal, highly viscous rotating flow. Thin thermal boundary layers are formed on the horizontal boundaries. When the thermal boundary layers from adjacent cells meet they separate from the horizontal boundary and form a thermal plume on the vertical boundary between cells. The body force in the plumes drives the viscous core flow. It is found that the Nusselt number for the total heat transfer between the horizontal boundaries is proportional to the Rayleigh number to the one-quarter power. Good agreement with experiment is obtained.

NOMENCLATURE

- A, normal gradient of the velocity on the horizontal boundary, equation (17);
- d, distance between the horizontal plates;
- g, acceleration of gravity;
- $\mathbf{j}$, unit vector in the vertical direction;
- k, thermal conductivity;
- m, integer $(=1, 3, 5, \ldots)$;
- Nu, Nusselt number;
- Nu_l, local Nusselt number, equation (23);
- n, integer $(=1, 3, 5, \ldots)$;
- P, hydrostatic pressure;
- Pr, Prandtl number, equation (2);
- p, pressure;
- q_w, local heat flux to the wall per unit area;
- Ra, Rayleigh number, equation (1);
- T, temperature;
- T_{w1}, temperature of the lower horizontal plate;
- T_{w2}, temperature of the upper horizontal plate;
- $\mathbf{u}$, velocity vector;
- u_x, horizontal component of velocity;
- u_y, vertical component of velocity;
- x, horizontal coordinate measured from the center of the cell;
- x_1, horizontal coordinate measured from the vertical boundary;
- y, vertical coordinate measured from the center of the cell;
- y_1, vertical coordinate measured from the horizontal boundary;
- $y_{1\delta}$, boundary-layer thickness.

Greek symbols

- α, coefficient of thermal expansion;
- β, temperature gradient in the absence of convection;
- γ, normal gradient of the velocity on the vertical boundary between cells;
- δ, half-width of a cell;
- θ, temperature difference with the isothermal core as reference;
- κ, thermal diffusivity;
- ν, kinematic viscosity;
- ρ, density;
- ψ, stream function;
- $'$, denotes dimensional variables.

1. INTRODUCTION

WHEN a fluid which is confined between two horizontal plates is heated from below, cellular convection occurs. The first observation of the phenomenon is attributed to Bénard [1]. A solution of the linearized stability problem which determines the onset of the convective motion was first given by Rayleigh [2]. Convective motion was predicted if the dimensionless parameter

$$Ra = \frac{\alpha\beta d^4 g}{\kappa\nu} \tag{1}$$

exceeds a critical value. The critical Rayleigh number is a function of the boundary conditions and for isothermal fixed boundaries its value is 1707. The extensive literature on the stability problem is summarized by Chandrasekhar [3]. The predicted onset of convection is in good agreement with experimental measurements.

Observations of cellular convection over a wide range of Rayleigh and Prandtl numbers have been carried out by Silveston [4]. At moderate Rayleigh numbers steady cellular convection is observed. The convective cells often take the form of two-dimensional rolls. For large Rayleigh numbers the convective flow becomes turbulent. Unlike the linear stability problem, the theory for steady cellular convection has received relatively little attention. A solution for steady cellular convection requires solving a set of coupled, nonlinear partial differential equations. Extensions of the linear theory into the nonlinear regime have been given by Malkus and Veronis [5], Kuo [6] and Platzman [7]. However, these theories are expansions about the critical Rayleigh number and are expected to be valid only when the Rayleigh number is near its critical value.

For steady cellular convection the Prandtl number

$$Pr = \frac{\nu}{\kappa} \tag{2}$$

is a governing parameter as well as the Rayleigh number. For large values of the Prandtl number diffusion of vorticity will occur much more rapidly than conduction of heat. In forced convection the result is that thermal boundary layers are thin compared with velocity boundary layers. In this paper a theory is given for cellular convection which is valid for large Prandtl and Rayleigh numbers. It is found that thermal gradients are restricted to thin layers on the boundaries of each cell. In the core of each cell is a highly viscous isothermal flow. The analysis is valid for laminar, two-dimensional cellular convection between horizontal plates.

2. BASIC EQUATIONS

In order to obtain solutions for steady cellular convection it is necessary to solve simultaneously the equations for conservation of mass, momentum, and energy. In writing the conservation equations the Boussinesq approximation is used, that is, the density and the coefficients (viscosity, thermal diffusivity, etc.) are assumed to be constant except for the density in the body force term in the momentum equation. A linear relation is assumed between the variations of temperature and density

$$\rho' - \rho_0' = -\rho_0'\alpha(T' - T_0') \tag{3}$$

where T_0' is the temperature at which $\rho' = \rho_0'$. Primes denote dimensional variables. Introducing $\theta' = T' - T_0'$ and $P' = p' + \rho_0' g y'$ the conservation equations for steady, laminar flows are

$$\nabla \mathbf{u}' = 0 \tag{4}$$

$$(\mathbf{u}' . \nabla')\mathbf{u}' = -\frac{1}{\rho_0'}\nabla' P' + \nu\nabla'^2\mathbf{u}' + \alpha\theta' g\mathbf{j} \tag{5}$$

$$(\mathbf{u}' . \nabla')\theta' = \kappa\nabla'^2\theta'. \tag{6}$$

The following nondimensional variables are introduced

$$\nabla = d\nabla', \qquad \mathbf{u} = \frac{\mathbf{u}'d}{\kappa},$$

$$P = \frac{P'd^2}{\rho_0'\nu\kappa}, \qquad \theta = \frac{\theta'}{\beta d}.$$

Using the thermal diffusivity in the dimensionless velocity is necessary for this problem. The reason is that the body force term in the momentum equation is balanced by the viscous term. Using the thermal diffusivity it will be found that the dimensionless velocity is a function only of the Rayleigh number. Substitution of these nondimensional variables into equations (4–6) gives

$$\nabla . \mathbf{u} = 0 \tag{7}$$

$$\frac{1}{Pr}(\mathbf{u} . \nabla)\mathbf{u} = -\nabla P + \nabla^2 \mathbf{u} + Ra\theta \mathbf{j} \tag{8}$$

$$(\mathbf{u} . \nabla)\, \theta = \nabla^2 \theta. \tag{9}$$

In this paper these equations are solved for a fluid confined between two horizontal plates at $y = \pm\frac{1}{2}$. Boundary conditions for the temperature and velocity are required on the horizontal plates. We require that $\theta = -\frac{1}{2}$ at $y = \frac{1}{2}$ and $\theta = +\frac{1}{2}$ at $y = -\frac{1}{2}$. The condition that there be no flow through the horizontal boundaries requires that $u_y = 0$ at $y = \pm\frac{1}{2}$. The no-slip boundary condition on the horizontal plates requires that $u_x = 0$ at $y = \pm\frac{1}{2}$.

3. CELLULAR CONVECTION MODEL

The fluid confined between the horizontal planes at $y = \pm\frac{1}{2}$ is divided into cellular two-dimensional rolls, alternate rolls flow in the clockwise and counterclockwise directions. The dividing planes between cells are at $x = \pm\delta$, $\pm 3\delta$, $\pm 5\delta$, ... For arbitrary values of the Prandtl and Rayleigh numbers the solution of equations (7–9) in a rectangle is still prohibitively difficult.

For large values of the Prandtl number, $Pr \gg 1$, the convection terms in the momentum equation may be neglected, as long as the dimensionless velocity is not $O(Pr)$, and equation (8) may be written

$$O = -\nabla P + \nabla^2 \mathbf{u} + Ra\theta \mathbf{j}. \tag{10}$$

When the kinematic viscosity is large compared to the thermal diffusivity the body forces associated with temperature gradients induce small velocities and convection terms can be neglected compared with viscous terms. For large values of the Prandtl number the following model for cellular convection will be hypothesized. The core of each rectangular cell is an isothermal, highly viscous rotating flow. On the hot and cold horizontal boundary plates are thin thermal boundary layers. When the two hot boundary layers from adjacent cells meet they separate from the lower boundary and form a hot plume which rises to the upper boundary along the vertical plane between cells. When this hot plume comes into contact with the upper cold surface a stagnation point thermal boundary layer is formed. As the flow splits and continues along the cold upper boundary the stagnation point boundary layer becomes the cold thermal boundary layers in the two adjacent cells. When the cold thermal boundary layers from adjacent cells meet they separate from the upper boundary and form a cold plume that descends to the lower surface. The temperature excess and temperature deficit in the plumes drive the viscous flow. The model is illustrated in Fig. 1.

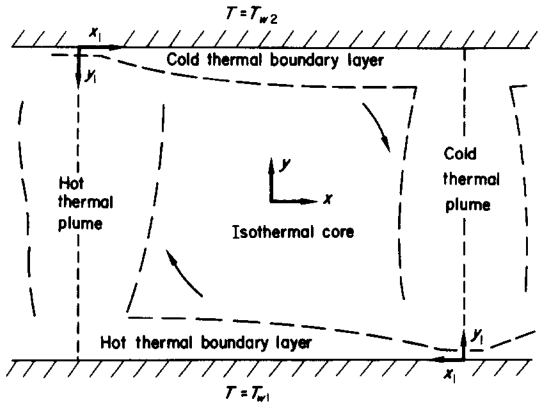

FIG. 1. Illustration of the boundary-layer model for cellular convection.

Actually there is a series of thermal layers as the boundary layers continue to convect in a spiral motion. However, the analysis given in this paper will be restricted to the first layer adjacent to the boundaries of the cell. The core is assumed to be isothermal. This should be a good approximation, since most of the temperature drop will occur in the first layer. That the boundary layers and plumes are in fact thin compared with the dimensions of the cell will be verified after a solution is obtained.

4. ISOTHERMAL CORE

Since the boundary layers and plumes are assumed to be thin, it is appropriate to obtain the two-dimensional core flow in a rectangular cell with dimensions 2δ and 1. Since the core is assumed to be isothermal, the energy equation is not required. By symmetry it is appropriate to take $\theta = 0$ in the isothermal core and the momentum equation, equation (10), reduces to

$$-\nabla P + \nabla^2 \mathbf{u} = 0. \tag{11}$$

Introducing the dimensionless stream function ψ, $u_x = -\partial\psi/\partial y$, $u_y = \partial\psi/\partial x$, equations (7) and (11) combine to give the biharmonic equation for the stream function

$$\nabla^4 \psi = 0. \tag{12}$$

This is the basic equation for flows in which the viscous forces dominate over the inertia forces.

The condition that there be no flow through the boundaries of the cell requires that $u_x = 0$ at $x = \pm\delta$ and $u_y = 0$ at $y = \pm\frac{1}{2}$. The viscous core flow is driven by the body force acting in the convective plumes. It will be shown that $(\partial u_y/\partial x)_{x=\pm\delta}$ is related to the integral of the temperature deficit (excess) in the convective plume. Since the integral of the temperature deficit in the plume is independent of y, it is appropriate to require $\partial u_y/\partial x = \gamma$ at $x = \pm\delta$ for the core solution where γ is a constant which will be determined from the plume solution.

In order to satisfy the above boundary conditions it is necessary that ψ be an even function in both x and y. By separation of variables it is found that

$$\psi = \sum_n [\cos \alpha_n y(A_n \cosh \alpha_n x + D_n x \sinh \alpha_n x)] + \sum_m [\cos \beta_m x(B_m \cosh \beta_m y + C_m y \sinh \beta_m y)] \tag{13}$$

is an even function in x and y which satisfies the biharmonic equation. In order to satisfy the boundary conditions given above it is necessary that

$$\alpha_n = n\pi, \qquad n = 1, 3, 5, 7, \ldots$$

$$\beta_m = m\pi/2\delta, \qquad m = 1, 3, 5, 7, \ldots$$

$$A_n = -\frac{2\gamma\delta}{n^2\pi^2} \sin\frac{n\pi}{2} \frac{\tanh n\pi\delta}{\cosh n\pi\delta}$$

$$D_n = \frac{2\gamma}{n^2\pi^2} \frac{\sin(n\pi/2)}{\cosh n\pi\delta}$$

$$C_m = -\frac{64\delta^2\gamma}{m^3\pi^3} \frac{\sin(m\pi/2)}{\sinh(m\pi/4\delta) + (m\pi/4\delta)\operatorname{sech}(m\pi/4\delta)} \sum_n \left(1 + \frac{4\delta^2 n^2}{m^2}\right)^{-2}$$

$$B_m = \frac{32\delta^2\gamma}{m^3\pi^3} \frac{\sin(m\pi/2)\tanh(m\pi/4\delta)}{\sinh(m\pi/4\delta) + (m\pi/4\delta)\operatorname{sech}(m\pi/4\delta)} \sum_n \left(1 + \frac{4\delta^2 n^2}{m^2}\right)^{-2}.$$

The velocity components within the viscous core are given by

$$u_x = \sum_n \left[\frac{2\gamma}{n\pi} \frac{\sin(n\pi/2)\sin(n\pi y)}{\cosh n\pi\delta}(x \sinh n\pi x - \delta \tanh n\pi\delta \cosh n\pi x)\right] + \sum_m \left[\frac{64\delta^2\gamma}{m^3\pi^3} \frac{\sin(m\pi/2)\cos(m\pi x/2\delta)}{\sinh(m\pi/4\delta) + (m\pi/4\delta)\operatorname{sech}(m\pi/4\delta)} \left(\left\{1 - \frac{m\pi}{4\delta}\tanh\frac{m\pi}{4\delta}\right\}\sinh\frac{m\pi y}{2\delta} + \frac{m\pi y}{2\delta}\cosh\frac{m\pi y}{2\delta}\right)\left(\sum_n \left\{1 + \frac{4\delta^2 n^2}{m^2}\right\}^{-2}\right)\right] \tag{14}$$

$$u_y = \sum_n \left[\frac{2\gamma}{n^2\pi^2} \frac{\sin(n\pi/2)\cos(n\pi y)}{\cosh n\pi\delta} \left(\left\{ 1 - n\pi\delta \tanh n\pi\delta \right\} \sinh n\pi x + n\pi x \cosh n\pi x \right) \right] + \sum_m \left[\frac{32\delta\gamma}{m^2\pi^2} \frac{\sin(m\pi/2)\sin(m\pi x/2\delta)}{\sinh(m\pi/4\delta) + (m\pi/4\delta)\,\mathrm{sech}\,(m\pi/4\delta)} \left(y \sinh\frac{m\pi y}{2\delta} - \frac{1}{2}\tanh\frac{m\pi}{4\delta}\cosh\frac{m\pi y}{2\delta} \right) \left(\sum_n \left\{ 1 + \frac{4\delta^2 n^2}{m^2} \right\}^{-2} \right) \right]. \quad (15)$$

Before further computations are carried out the dimensions of the rectangular cell, 1 and δ, will be specified. It is assumed that the cell size is the same as that given by the linear stability theory. That is $\delta = 0{\cdot}504$ (see Chandrasekhar [3]). Alternative methods of determining δ may be used. For example, a minimum energy principal or a separation condition for the horizontal boundary layer could be used.

Taking $x = \pm\delta$ and substituting the above value for δ equation (15) can be used to determine the vertical component of the velocity on the boundary between cells. The ratio u_y/γ at $x = \pm 0{\cdot}504$ is plotted against y in Fig. 2. The

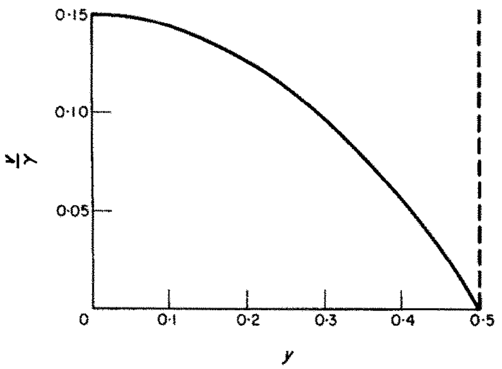

FIG. 2. Dependence of the ratio v/γ evaluated on the boundary between cells on y.

mean value of the vertical velocity on the boundary between cells is

$$u_y = \pm 0{\cdot}1006\gamma. \quad (16)$$

In order to determine the structure of the thermal boundary layers the normal gradient of the velocity on the horizontal boundaries will be required. We define

$$A = \left(\frac{\partial u_x}{\partial y} \right)_{y=\pm\frac{1}{2}}. \quad (17)$$

Taking the derivative of equation (14) and substituting $y = \pm\frac{1}{2}$ the dependence of A on x can be determined. The ratio A/γ is plotted against x in Fig. 3. The mean value of the

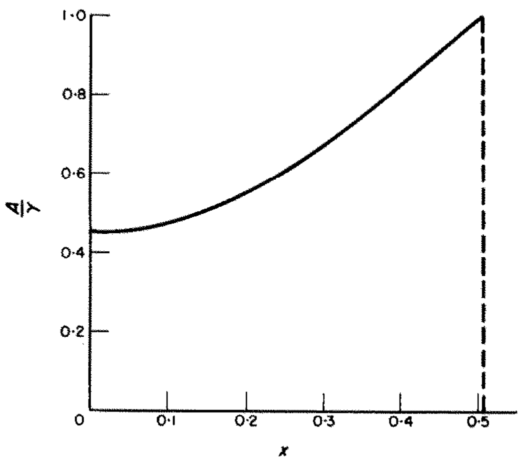

FIG. 3. Dependence of the ratio A/γ evaluated on the horizontal boundaries on x.

normal gradient of the velocity on the horizontal boundaries is

$$A = 0{\cdot}634\,\gamma. \quad (18)$$

A good empirical fit to the dependence of A on x is given by

$$A = \left(1 - 0{\cdot}552 \sin\frac{\pi x_1}{2\delta} \right)\gamma. \quad (19)$$

Equations (14) and (15) may be used to determine the velocity distribution throughout the isothermal core.

5. THERMAL BOUNDARY LAYERS

The velocity distribution obtained from the core solution may be used to determine the temperature distribution in the thermal boundary layers on the horizontal boundaries. Since it is postulated that the thermal boundary layers are thin, $y_{1\delta} \ll 1$, it is appropriate to assume a linear velocity profile to be valid within the thermal boundary layers,

$$u_x = Ay_1 \tag{20}$$

where A has been defined in equation (17) and y_1 is the distance from the horizontal boundary. To simplify the analysis we will take A to be a constant equal to its mean value on the horizontal boundary as given in equation (18). Experience with forced convection boundary layers shows that for the variation of A given in Fig. 3 this approximation should not lead to serious errors. With A = constant it is appropriate to take $u_y = 0$ in the thermal boundary layers.

Having prescribed the velocity only the energy equation is required to obtain the temperature distribution in the thermal boundary layers. With the velocity distribution as given above and the boundary layer form of equation (9), $\partial^2/\partial y^2 \gg \partial^2/\partial x^2$, we have

$$Ay\frac{\partial \theta}{\partial x_1} = \frac{\partial^2 \theta}{\partial y_1^2} \tag{21}$$

where x_1 is measured from the origin of the thermal boundary layer. The boundary conditions for the boundary layer solution of equation (21) are

$$\theta \to 0 \quad \text{as} \quad y_1 \to \infty$$

and

$$\theta = \tfrac{1}{2} \quad \text{at} \quad y_1 = 0.$$

A similarity solution of equation (21) which satisfies these boundary conditions is

$$\theta = \tfrac{1}{2}\Big[1 - 0{\cdot}537 \int_0^{y_1(A/x_1)^{1/3}} \exp(-z^3/9)\,dz\Big] \tag{22}$$

This solution is invalid near the vertical boundary where the stagnation point flow must be taken into account. For the dependence of A on x given in Fig. 3 it is expected that the stagnation point solution will be required only in the immediate vicinity of the vertical boundary and that equation (22) will be valid over a large fraction of the horizontal boundary. Therefore the origin of the thermal boundary should be near the vertical boundary between cells and it is a good approximation to measure x_1 from the vertical boundary.

The local heat transfer to the horizontal boundary can be expressed in terms of a local Nusselt number

$$Nu_l = \frac{q'_w d}{k(T_{w2} - T_{w1})} \tag{23}$$

where q'_w is the local heat flux per unit area. The local Nusselt number obtained from equation (22) is

$$Nu_l = 0{\cdot}268\left(\frac{A}{x_1}\right)^{1/3}. \tag{24}$$

The thickness of the thermal boundary layer, $y_{1\delta}$, is defined as the distance from the boundary to where $\theta = 0{\cdot}05$. The maximum boundary-layer thickness is at $x_1 = 2\delta = 1{\cdot}008$ and from equation (22) it is found to be

$$y_{1\delta} = 2{\cdot}08\left(\frac{x_1}{A}\right)^{1/3} = 2{\cdot}08\,A^{-1/3}. \tag{25}$$

Thermal boundary layers in fluids with large Prandtl number have been considered in detail by Lighthill [8]. He has solved the problem considered above for arbitrary dependence of A on x. For constant wall temperature the local Nusselt number obtained by Lighthill is

$$Nu_l = 0{\cdot}268\,A^{1/2}\Big(\int_0^x A^{1/2}\,dx\Big)^{-1/3}. \tag{26}$$

The results given above are valid for the boundary layers on both the horizontal plates.

6. THERMAL CONVECTIVE PLUMES

It is now possible to obtain the temperature distributions in the convective plumes. The centerline of each plume is in fact the division between the adjacent cells. However, it is convenient to determine the structure of an entire plume which in fact belongs to two adjacent cells. Since it is hypothesized that the plumes are thin it is appropriate to take u_y to be independent of x within the plumes. It will also be assumed that u_y is independent of y and is equal to its mean value on the vertical boundary between cells as given in equation (16). Therefore we take u_y = constant and $u_x = 0$ in the solution for the temperature distribution in the plumes. These approximations allow an analytic expression for the temperature distribution in the plume to be obtained. It will be shown that the velocities within the cells and the heat flux to the boundaries are not dependent on the temperature distribution in the plumes.

Since the velocity distribution in the thermal plumes is prescribed only the energy equation is required to obtain the temperature distribution in the thermal plumes. The governing boundary layer form of the energy equation is the same as for the thermal boundary layers except that x and y are interchanged,

$$u_y \frac{\partial \theta}{\partial y_1} = \frac{\partial^2 \theta}{\partial x_1^2} \qquad (27)$$

where y_1 is the distance from the horizontal boundary where the plume is formed and x_1 is the distance from the centerline of the plume. However, the boundary conditions for the plume structure differ from those for the thermal boundary-layer structure. In the boundary-layer solution of equation (27) the required boundary conditions for the plume structure are that $\theta \to 0$ as $x_1 \to \pm\infty$ and that the initial temperature distribution, at $y_1 = 0$, is specified. The solution of the heat equation with these boundary conditions is known as Laplace's solution and is given by (see Carslaw and Jaeger [9])

$$\theta = \frac{1}{2}\left(\frac{u_y}{\pi y_1}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \theta_0 \exp\left[-\frac{(x_1 - x')^2 u_y}{4 y_1}\right] \mathrm{d}x' \qquad (28)$$

where θ_0 is the initial temperature distribution at $y_1 = 0$.

Since the thermal plumes are formed from the separated thermal boundary layers, it is appropriate to set the initial temperature distribution in the plumes equal to the temperature distribution in the thermal boundary layers adjacent to the base of the plume. The initial temperature distribution in the plume is obtained from equation (22) by setting $x_1 = 2\delta = 1{\cdot}008$ and interchanging y_1 with x_1 with the result

$$\theta_0 = \tfrac{1}{2}\Big[1 - 0{\cdot}537 \int_0^{|x_1|(A/1\cdot 008)^{\frac{1}{3}}} \exp(-z^3/9)\,\mathrm{d}z\Big] \qquad (29)$$

Substitution of equation (29) into equation (28) and changing the variables of integration gives

$$\theta = \frac{1}{4}\left(\frac{1}{\pi y_1}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \Big[1 - 0{\cdot}537 \int_0^{(A/1\cdot 008)^{\frac{1}{3}}|\xi|/u_y^{\frac{1}{2}}} \exp(-z^3/9)\,\mathrm{d}z\Big] \exp\left[-\frac{(x_1 u_y^{\frac{1}{2}} - \xi)^2}{y_1}\right] \mathrm{d}\xi \qquad (30)$$

From equation (30) the temperature distribution in each convective plume can be determined.

7. MATCHING OF SOLUTIONS

The magnitude of the velocities in the isothermal core is proportional to γ. This constant will now be evaluated by relating it to the body force acting in the convective plumes. Although in the core solution γ was taken to be the normal gradient of the velocity on the vertical boundary, in terms of the plume structure it is the normal velocity at the outer edge of the plume. In order to determine the appropriate boundary condition on $\partial u_y/\partial x$ at the outer edge of the plume, we write the y component of the momentum equation valid within the thermal plume from equation (10),

$$\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} = +\frac{\partial P}{\partial y} - Ra\theta. \quad (31)$$

Within the thin convective plume it is consistent to neglect both $\partial^2 u_y/\partial y^2$ and $\partial P/\partial y$ compared with $\partial^2 u_y/\partial x^2$ and equation (31) reduces to

$$\frac{\partial^2 u_y}{\partial x^2} = -Ra\theta. \quad (32)$$

Integrating equation (32) noting that $\partial u_y/\partial x = 0$ on the centerline of the plume, we obtain

$$\gamma = \left(-\frac{\partial u_y}{\partial x_1}\right)_{x_1=\infty} = Ra\int_0^\infty \theta\, \mathrm{d}x_1. \quad (33)$$

It is seen that γ is proportional to the integral of the temperature deficit (excess) in the plume. Since there is no heat addition to the plume, the heat content of the plume is constant and the integral in equation (33) is independent of y and therefore γ is a constant. The initial temperature distribution at $y_1 = 0$ can be used to evaluate γ so that the evaluation of γ does not depend upon the plume structure. Substitution of equation (29) into equation (33) and integrating gives

$$\gamma = 0{\cdot}535\frac{Ra}{A^{\frac{1}{3}}}. \quad (34)$$

Substituting equation (18) into equation (34) and solving for γ gives

$$\gamma = 0{\cdot}701\, Ra^{\frac{3}{4}}. \quad (35)$$

Substituting equation (35) into equation (18) gives the mean value for A in terms of the Rayleigh number

$$A = 0{\cdot}444\, Ra^{\frac{3}{4}}. \quad (36)$$

Substituting equation (35) into equation (16) gives the mean value of the dimensionless vertical velocity on the boundary between cells in terms of the Rayleigh number

$$u_y = 0{\cdot}0705\, Ra^{\frac{3}{4}}. \quad (37)$$

Since the dimensionless velocities associated with the cellular convection are proportional to γ, they are therefore proportional to the Rayleigh number to the three-quarter power.

The local heat flux to the boundaries may now be related to the Rayleigh number. The local Nusselt number for the thermal boundary layers is obtained by substituting equation (36) into equation (24) with the result

$$Nu_l = 0{\cdot}204\frac{Ra^{\frac{1}{4}}}{x_1^{\frac{1}{3}}}. \quad (38)$$

An alternative expression for the local Nusselt number is obtained by substituting equations (19) and (35) into equation (26) with the result

$$Nu_l = 0{\cdot}238\, Ra^{\frac{1}{4}}\left(1 - 0{\cdot}552\sin\frac{\pi x_1}{2\delta}\right)^{\frac{1}{2}} \times\left[\int_0^x \left(1 - 0{\cdot}552\sin\frac{\pi x_1}{2\delta}\right)^{\frac{1}{2}} \mathrm{d}x\right]^{-\frac{1}{3}}. \quad (39)$$

The Nusselt number for the total heat transfer between the horizontal surfaces is obtained by taking the mean value of the local Nusselt number over the cell. Assuming equation (38) to be valid over the entire horizontal boundary, the total Nusselt number is

$$Nu = \frac{1}{2\delta}\int_0^{2\delta} Nu_l\, \mathrm{d}x_1 = 0{\cdot}304\, Ra^{\frac{1}{4}}. \quad (40)$$

Assuming equation (39) to be valid over the entire horizontal boundary, the total Nusselt number is

$$Nu = 0{\cdot}306\, Ra^{\frac{1}{4}}. \quad (41)$$

It is seen that the more exact theory given by Lighthill differs from the approximate theory by less than one per cent. It is expected that the relation for the heat transfer between the horizontal plates should be valid for large values of the Prandtl and Rayleigh numbers as long as the cellular convection is laminar.

The dimensionless maximum thickness of the thermal boundary layers is obtained by substituting equation (36) into equation (25),

$$y_{1\delta} = \frac{2{\cdot}72}{Ra^{\frac{1}{4}}}. \quad (42)$$

Since $y_{1\delta}$ is the ratio of the actual thickness of the thermal boundary layer to the plate separation, $y_{1\delta}$ must be small for the boundary-layer hypothesis to be valid. From equation (38) it is seen that the boundary-layer hypothesis is valid for large values of the Rayleigh number.

Another approximation that should be verified is the validity of dropping the convection terms in equation (8). From equation (37) we see that the dimensionless velocities are of the order $0{\cdot}1\ Ra^{\frac{1}{2}}$ and are large for large Ra. Therefore dropping the convection terms in equation (8) actually requires that $0{\cdot}1\ Ra^{\frac{1}{2}}/Pr \ll 1$.

8. COMPARISON WITH EXPERIMENT

Measurements of the heat transfer between horizontal plates for a wide range of Prandtl and Rayleigh numbers have been obtained by Silveston [4]. Of particular interest for comparison with the theory given in this paper are the measurements carried out with silicon oil AK350 which had an average Prandtl number of 3000 for Rayleigh numbers from 500 to 30000 and the measurements carried out with glycol which had an average Prandtl number of 130 for Rayleigh numbers from 1000 to 80000. The plate spacing used in these measurements ranged from 3 to 13 mm.

The approximations used in this paper are certainly valid for the silicon oil measurements and are marginally valid for the glycol experiments. It should be noted that Silveston found that the dependence of Nusselt number on Rayleigh number was virtually independent of the Prandtl number for Prandtl numbers from 1 to 3000. Of course, the theory given here is valid only for large Prandtl numbers so cannot explain the measurements for Prandtl numbers of order one. For intermediate Rayleigh numbers, $4000 < Ra < 44000$, Silveston correlates his data with the empirical relation

$$Nu = 0{\cdot}24\ Ra^{\frac{1}{4}} \qquad (43)$$

independent of Prandtl number. In this range of Rayleigh numbers the cellular convection is well developed and laminar. In the lower part of the range well defined, two-dimensional rolls were observed. As the Rayleigh number was increased the convection pattern became somewhat irregular. At values of the Rayleigh number below 4000 the cells are just being formed and the boundary-layer theory is not expected to be valid. At Rayleigh numbers greater than 44000 the cellular flow was observed to be turbulent and the laminar theory is not applicable.

It is appropriate to compare the empirical correlation, equation (43), with theoretical values for the total Nusselt number obtained from the boundary-layer theory, either equation (40) or equation (41). It is seen that the experimental dependence of the Nusselt number on the Rayleigh number is in agreement with the boundary-layer theory. The theoretical values for the constant of proportionality are somewhat larger than the experimental value. If the cell size was left as a free parameter then exact agreement between theory and experiment could be obtained. This difference may also be attributed to the approximations in the boundary-layer theory or to experimental errors. In particular it is difficult to maintain the constant temperature boundary conditions on the horizontal plates of the apparatus. A direct comparison of equation (41) with the measurements of Silveston [4] is given in Fig. 4.

The extensions of the linear theory into the

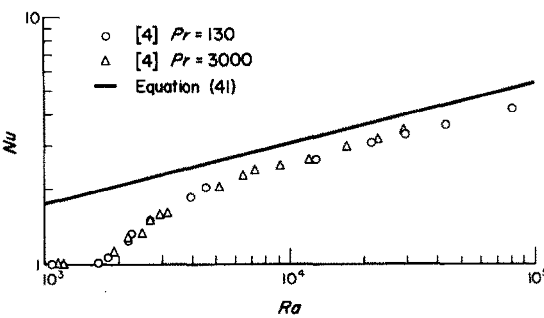

FIG. 4. Comparison of the boundary-layer theory for cellular convection with the heat-transfer measurements of Silveston [4].

nonlinear regime also predict the dependence of Nusselt number on Rayleigh number. The theory of Malkus and Veronis [5] gives results which appear to be convergent for Rayleigh numbers as large as ten times the critical Rayleigh number, however their results are for free surface boundary conditions so that a direct comparison with the experimental values is inappropriate. The theory of Platzman [7] appears to agree well with experiment for Rayleigh numbers up to 40000 for Prandtl numbers of order unity but the theory diverges from experiment for large Prandtl numbers.

Considering the number of approximations that have been included in the analysis the agreement between theory and experiment must be considered satisfactory. It is concluded that the boundary-layer theory is applicable and can predict the velocity and temperature distributions in Bénard cells for large values of the Prandtl and Rayleigh numbers.

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Résumé—Une théorie pour la convection cellulaire permanente à amplitude finie est exposée. La théorie demande que le nombre de Prandtl soit grand devant l'unité et que le nombre de Rayleigh soit élevé devant le nombre de Rayleigh critique; on considère seulement des cellules laminaires bidimensionnelles. Le noyau de chaque cellule est un écoulement en rotation fortement visqueux et isotherme. Des couches limites thermiques minces se forment sur les frontières horizontales. Lorsque les couches limites thermiques des cellules se rencontrent, elles se séparent de la frontière horizontale et forment un panache thermique sur la frontière verticale entre les cellules. La force volumique dans les panaches met en mouvement l'écoulement visqueux du noyau. On trouve que le nombre de Nusselt pour le flux total de chaleur entre les frontières horizontales est proportionnel au nombre de Rayleigh élevé à la puissance $\frac{1}{4}$. Un bon accord avec l'expérience est obtenu.

Zusammenfassung—Für stationäre Zellularkonvektion mit endlicher Amplitude wird eine Theorie entwickelt. Die Theorie verlangt, dass die Prandtl-Zahl gross gegen eins ist und dass die Raleigh-Zahl gross gegen die kritische Raleigh-Zahl ist. Nur zwei-dimensionale laminare Zellen werden betrachtet. Der Kern jeder Zelle enthält eine isotherme, hoch viskose rotierende Strömung. Dünne thermische Grenzschichten werden an den waagerechten Begrenzungen gebildet. Wo sich die thermischen Grenzschichten benachbarter Zellen treffen, lösen sie sich von der waagerechten Begrenzung und bilden eine thermische Auftriebszone entlang der senkrechten Begrenzung zwischen den Zellen. Die Auftriebskräfte in den Zonen treiben die zähe Kernströmung. Man findet, dass die Nusselt-Zahl für den Wärmetransport zwischen den waagerechten Platten proportional der vierten Wurzel der Raleigh-Zahl ist. Gute Übereinstimmung mit Versuchen wird erhalten.

Аннотация—Разработана теория стационарной ячеистой конвекции конечной амплитуды. Теория применима при числах Прандтля, значительно больших единицы, и чисел Релея, выше критических; рассматриваются только двумерные ламинарные ячейки. Ядро каждой ячейки-изотермическое, высоковязкое ротационное течение, а на горизонтальных границах образуются тепловые пограничные слои. При соприкосновении тепловых пограничных слоев соседних ячеек, они отделяются от горизонтальной границы и образуют тепловой восходящий поток на вертикальной границе между ячейками. Движущей силой вязкого ядра течения является объемная сила в области, занятой восходящим потоком. Установлено, что число Нуссельта для суммарного теплообмена между горизонтальными границами пропорционально числу Релея в степени $\frac{1}{4}$. Получено хорошее согласование с экспериментом.